DISCRETE MATHEMATICS AND ITS APPLICATIONS IN NETWORK ANALYSIS

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Abstract: In this article we will give a small introduction to the discrete mathematics and its application in network analysis. Because of limitations regarding this article (extended summary) and because we are aware that the scientific study of networks, such as computer networks, biological networks, and social networks, is an interdisciplinary field that combines ideas from mathematics, physics, biology, computer science, the social sciences, and many other areas, we will try to describe some basic types of networks studied by presentï day science as well as some techniques used to determine their structure.

Key words: Networks, discrete mathematics, network analysis, ViSOC

DISKRETNA MATEMATIKA I NJENE PRIMJENE U MREŽNOJ ANALIZI

Sažetak: U ovom radu dat Ĩemo samo kraĨi uvod u diskretnu matematiku te njezine primjene/aplikacije u mreġnoj analizi. BuduĨi smo svjesni ograniļenja vezanih za ovaj rad (proġireni saġetak), te buduĨi smo svjesni ļinjenice kako je znanstveno prouļavanje/studija mreġa, kao na primjer raļunalnih mreġa, bioloġkih mreġa, te druġtvenih mreġa, u biti interdisciplinarno polje u kojem se kombiniraju i isprepliĨu ideje iz matematike, fizike, biologije, raļunalnih znanosti, i druġtvenih znanosti, kao i mnogih drugih podruļja, pokuġat Ĩemo opisati neke osnovne tipove mreġa koji su bili predmetom prouļavanja znanosti do danaġnjeg vremena, kao i neke tehnike koje se koriste za utvrĽivanje njihove strukture.

Ključne riječi: Mreġe, diskretna matematika, mreġna analiza, ViSOC

1. INTRODUCTION

A network is, in its simplest form, a collection of points joined together in pairs by lines. Networks are thus a general yet powerful means of representing patterns of connections or interactions between the parts of a system. In the jargon of the field the points are referred to as *vertices* or *nodes* and the lines are referred to as *edges*. Many objects of interest in the physical, chemical, biological and social sciences can be thought of as networks and thinking of them in this way can often lead to new and useful insights [Newman, 2010].

Scientists in a wide variety of fields have, over the years, developed an extensive set of tools ï mathematical, computational, and statistical ï for analyzing, modeling and understanding networks. Many of these tools start from a simple network representation, a set of vertices and edges, and after suitable calculations tell you something about the network that might be useful for you: which is the best connected vertex, say, or the length of a path from one vertex to another. Other tools take the form of network models that can make mathematical predictions about processes taking place on networks, such as the way traffic will flow over Internet or the way a disease will spread through a community. Because they work with network in their abstract form, these tools can in theory be applied to almost any system represented as network. Thus if there is a system you are interested in, and it can usefully be represented as a network, then there are hundreds of different tools out there, already developed and well understood, that you can immediately apply to the analysis of your system. Certainly not all of them will give useful results ï which measurements or calculations are useful for a particular system depends on what the system is and does and on what specific questions you are trying to answer about it.

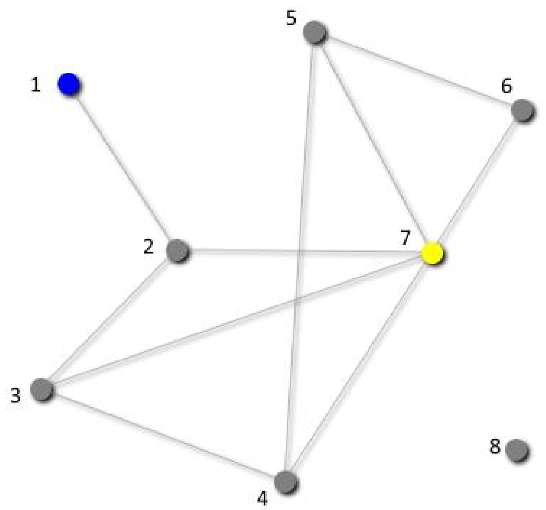

Figure 1. A small network composed of eight vertices and ten edges

In this article we introduce the basic theoretical tools used to describe and analyze networks, most of which come from graph theory or better to say from discrete mathematics, the branch of mathematics that deals with networks. To begin at the beginning, a network ï also called a graph in the mathematical literature ï is, as we have said, a collection of vertices joined by edges. Vertices and edges are also called nodes and links in computer science, sites and bonds in physics, and actors and ties in sociology. Table 1 gives some examples of vertices and edges in particular networks [Newman, 2010], [Kumar and others, 2000].

Table 1. Vertices and edges in networks

Network	Vertex	Edge
Internet	Computer or router	Cable or wireless data connection
World Wide Web	Web page	Hyperlink
Citation Network	Article, patent or legal case	Citation
Power grid	Generating station substation	Transmission line
Friendship network	Person	Friendship
Metabolic network	Metabolite	Metabolic reaction
Neural network	Neuron	Synapse
Food web	Species	Predation

2. SOME EXAMPLES OF NETWORKS

Throughout this article we will normally denote the number of vertices in a network by *n* and the number of edges by *m*, which is common notation in the mathematical literature. There are a number of different ways to represent a network mathematically. A usual representation of a network for present purposes is the *adjacency matrix*. The adjacency matrix **A** of a simple graph is the matrix with elements *Aij* such that

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices } i \text{ and } j, \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in \{1, 2, \ldots n\}. \tag{1}$$

For example, the adjacency matrix of the network on Fig. 1 is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

Two points to notice about the adjacency matrix are that, first, for a network with no selfï edges such as this one the diagonal matrix elements are all zero, and second that it is symmetric, since if there is an edge between *i* and *j* then there is an edge between *j* and *i*.

Many of the networks in science and theory have edges that form simple on/off connections between vertices. Either they are there or they are not. In some situations, however, it is useful to represent edges as having a strength, weight, or value to them, usually a real number. Thus in the Internet edges might have weights representing the amount of data flowing along them or their bandwidth. In a food web predatorï prey interactions might have weights measuring total energy flow between prey and predator. In a social network connections might have weights representing frequency of contact between actors. Such *weighted* or *valued networks* can be represented by giving the elements of the adjacency matrix values equal to the weights of the corresponding connections. Thus the adjacency matrix

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{pmatrix} \qquad (3)$$

represents a weighted network in which the connection between vertices 1 and 2 is twice as strong as that between 1 and 3, which in turn is twice as strong as that between 2 and 3.

Last application of discrete mathematics regarding network analysis in this article will be a *directed networks*. A directed network or *directed graph*, also called a *digraph* for short, is a network in which each edge has a direction, pointing *from* one vertex *to* another. Such edges are themselves called *directed edges*, and can be represented by lines with arrows on them ï see Fig. 2. Some of examples of directed networks are World Wide Web, in which hyperlinks run in one direction from one web page to another, food webs, in which energy flows from prey to predators, and citation networks, in which citations point from one paper to another. The adjacency matrix of a directed network has matrix elements

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge } \textit{from } j \textit{ to } i, \\ 0 & \text{otherwise} \end{cases}, \qquad i, j \in \{1, 2, \ldots n\}. \qquad (4)$$

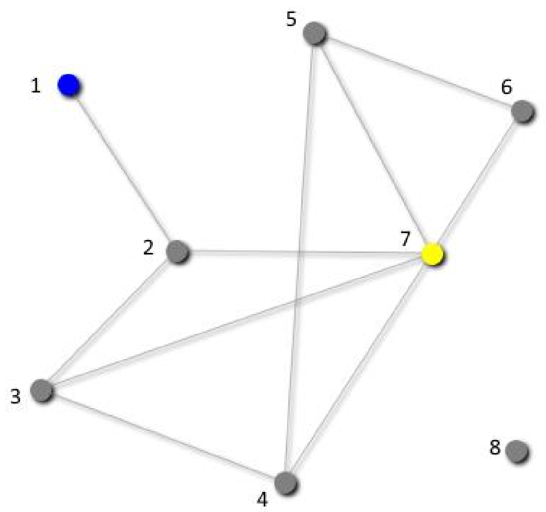

Figure 2. A directed network

Notice the direction of the edge here ï it runs *from* the second index *to* the first. This is slightly counterï intuitive, but it turns out to be convenient mathematically and it is the convention we adopt in this article. Next, note that this matrix is not symmetric. In general the adjacency matrix of directed network is asymmetric.

3. CONCLUDING REMARKS

There are many systems of interest to scientists that are composed of individual parts or components linked together in some way. Examples include the Internet, a collection of computers linked by data connections, and human societies, which are collections of people linked by acquaintance or social interaction. Many aspects of these systems are worthy to study. The pattern of connections in a given system can be represented as a network, the components of the system being the network vertices and the connections the edges. Upon reflection it should come as no surprise that the structure of such networks, the particular pattern of interactions, can have a big effect on the behavior of the system [Newman, 2010]. The pattern of connections between computers on the Internet, for instance, affects the routes that data take over the network and the efficiency with which the network transports those data. The connections in a social network affect how people learn, from opinions, and gather news, as well as affecting other less obvious phenomena, such as the spread of disease. Unless we know something about the structure of these networks, we cannot hope to understand fully how the corresponding system works.

The final aim of our work is to develop a new/unique and efficient tool dedicated to the analysis and visualization of community structure in small networks of various kinds, with an accent on biological and social networks. In this moment a simple version of our tool (ViSOC) existing. ViSOC is a free tool to share, and it is made in Microsoft Visual Studio (C#). You can freely copy, distribute, transmit, and use this tool for scientific and nonï commercial purposes. But no modification of the contents is allowed. Next, under the same terms of above licence, it is possible to share your ideas or remarks with our team, and so contribute them to our tool. Our goal is to help researchers to create exciting, interactive, engaging and rich visualization to test or analyze their examples regarding realï world networks.

LITERATURE

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